Non-Abelian gauge theory for quantum Heisenberg antiferromagnetic chain

1994 J. Phys. A: Math. Gen. 27 4695

(http://iopscience.iop.org/0305-4470/27/13/042)

J. Phys. A: Math. Gen. 27 (1994) 4695–4708. Printed in the UK

Non-Abelian gauge theory for quantum Heisenberg antiferromagnetic chain

Chigak Itoi† and Hisamitsu Mukaida‡

† Department of Physics and Atomic Energy Research Institute, College of Science and Technology, Nihon University Kanda Surugadai, Chiyoda-ku, Tokyo 101, Japan
‡ Institute of Physics, College of Liberal Arts and Sciences, University of Tokyo, Meguro-ku, Komaba, 153 Japan

Received 17 November 1993

Abstract. A continuum $SU(2S) \times U(1)$ gauge theory for a Heisenberg antiferromagnet with arbitrary spin S is obtained. We show that the level-1 $SU(2)$ WZW model with a certain marginally irrelevant perturbation describes the low-energy phenomena of the Heisenberg model with spin-$\frac{1}{2}$. The model with spins other than $\frac{1}{2}$ can be described in the level-2S $SU(2)$ WZW model with a certain relevant perturbation. We give a qualitative explanation of the Haldane conjecture for Heisenberg antiferromagnets.

1. Introduction

Several years ago, Haldane [1] obtained a quite surprising result for one-dimensional (1D) quantum Heisenberg antiferromagnets. Since then, many experimental, numerical and theoretical physicists have been involved in studying 1D quantum antiferromagnets because of this 'Haldane conjecture'. His claim is as follows: the half-odd-integer-spin Heisenberg model has gapless excitations on a unique ground state, while the integer-spin Heisenberg model only has excitations with a gap on a unique ground state. He obtained this result on the basis of a large spin approximation in terms of spin coherent states. Then he derived the $O(3)$ nonlinear sigma model with the θ term at $\theta = 2S\pi$ as the low-energy effective theory for the spin-S Heisenberg antiferromagnet. The $\boldsymbol{n}$-field in the $O(3)$ nonlinear sigma model is the block spin variable $S_{2x} - S_{2x+1}$ which can describe the low-energy physics of the antiferromagnet. The θ term is induced into the effective action of the $\boldsymbol{n}$-field by an integration of the short-wave variable $S_{2x} + S_{2x+1}$. We can regard the θ term as a Berry phase which is a first-order derivative of a long-wave variable in the effective action. Since the θ term is quantized by $2\pi \mathrm{i} S$, it can affect the physics only in a half-odd-integer spin case. The θ term tells us that the Heisenberg model must be classified into two universality classes: half-odd-integer and integer spins. It is well known that the $O(3)$ nonlinear sigma model without the θ term has only a disorder phase and the spin-$\frac{1}{2}$ Heisenberg model has gapless excitations in Bethe's exact solution. Therefore we understand the Haldane conjecture: it holds exactly in the spin-$\frac{1}{2}$ case by the Bethe ansatz solution; for other spins, however, there are no exact proofs of the Haldane conjecture.

The most remarkable study enabling theoretical understanding of the 'Haldane gap' was performed by Affleck *et al* [2]. They studied a slightly modified spin-1 model which has an exactly solvable ground state and only gap excitations. Since this stimulating work was published spin-1 chains have been studied extensively both in theoretical and experimental physics. The results in all the studies agree with the Haldane conjecture.

There are also many works in continuum-field-theory approaches to this problem. Continuum field theories have many advantages for calculating various quantities in low-energy physics although they are not suitable for rigorous discussions. Since the Heisenberg antiferromagnet is believed to be an effective theory for the half-filled multi-band Hubbard model with strong Hunt-rule coupling, Affleck and Haldane [3] studied the Hubbard model using continuum field theory instead of the Heisenberg model. Low-energy phenomena in the Hubbard model can be described by a relativistic field theory of Dirac fermions $\psi_{i\alpha x}$, $(i = 1, \ldots, 2S, \alpha = \uparrow, \downarrow)$ where i is an $SU(2S)$ colour index and α is an $SU(2)$ spin index. The Hunt-rule coupling is rewritten into several four-fermion interactions which are either marginally relevant or marginally irrelevant operators. Using a bosonization technique and a renormalization-group analysis, they concluded that the low-energy excitation in the spin-S Heisenberg model is described by the level-$2S$ $SU(2)$ WZW model with some perturbations. Excitations with respect to $SU(2S)$ colour and $U(1)$ degrees of freedom cannot survive in the low-energy limit due to relevant interactions. In the spin-$\frac{1}{2}$ case, the consistency of this result with the Bethe ansatz solution is checked using the finite-size scaling method [4, 5]. This agreement implies that their argument is plausible despite the complicated derivation.

In this paper, we would like to give a continuum-field-theory approach to the Heisenberg model. We employ $SU(2S) \times U(1)$ gauge theory in two dimensions to describe the spin-S Heisenberg antiferromagnet in one dimension. The argument in this method becomes much simpler than that of Affleck and Haldane, since we can deal directly with the Heisenberg model without passing through the Hubbard model. Confinement to $SU(2S) \times U(1)$ degrees of freedom is achieved by use of the $SU(2S) \times U(1)$ gauge field and the remaining $SU(2)$ spin excitations are described by the level-$2S$ $SU(2)$ WZW model. Our result is totally consistent with that of Affleck and Haldane.

This paper is organized as follows. In section 2, we derive a continuum gauge theory for the Heisenberg model. In section 3, we discuss the continuum gauge theory without four-fermion interactions, for which the operator contents of this gauge theory are given. We show that this model is identical to the level-$2S$ $SU(2)$ WZW model. In section 4, we study the spin-$\frac{1}{2}$ Heisenberg model by means of $U(1)$ gauge theory with certain four-fermion interactions. Our result is justified by the Bethe ansatz solution. In section 5, the qualitative features of the Heisenberg model with general spin S are discussed in the perturbed WZW model. We will see how discrimination between the two universality classes of half-odd-integer and integer spin Heisenberg model occurs. In the last section, we apply our effective gauge theory to integrable antiferromagnetic chains with spin-1 [6, 7].

2. Continuum gauge theory for the Heisenberg model

The spin-S Heisenberg model can be written in terms of electron operators $C_{\alpha i x}$, $\alpha = \uparrow, \downarrow$, $i = 1, \ldots, 2S$, where α, i and x are spin, colour and site indices respectively. The spin operator S_x on one site x can be represented by

$$S_x = C^{\dagger}_{\alpha i x}(\sigma_{\alpha\beta}/2)C_{\beta i x} \tag{2.1}$$

where the $\sigma_{\alpha\beta}$ are the Pauli matrices. The following constraints on the physical states are necessary for the correct spin-S representation

$$C^{\dagger}_{\alpha i x}C_{\alpha i x}|\mathrm{phys}\rangle = 2S|\mathrm{phys}\rangle \tag{2.2}$$

$$C^{\dagger}_{\alpha i x}\tau^{a}_{ij}C_{\alpha j x}|\mathrm{phys}\rangle = 0 \tag{2.3}$$

where τ^a is an algebra of $SU(2S)$. Equation (2.2) is an electron number constraint and equation (2.3) is a singlet condition with respect to the colour index on each site. $\boldsymbol{S}_x$ satisfies a spin commutation relation on the constrained states. In this representation, the Hamiltonian is

$$H = \sum_{\langle xy \rangle} \boldsymbol{S}_x \cdot \boldsymbol{S}_y = -\frac{1}{2} \sum C^{\dagger}_{\alpha i x} C_{\alpha j y} C^{\dagger}_{\beta j y} C_{\beta i x} + \text{constant} \tag{2.4}$$

which has $SU(2S) \times U(1)$ invariance on each site.

We are going to obtain a continuum field theory for this model. First we treat this model in a mean-field approximation and next take into account the correction to this approximation. Finally, we obtain a low-energy effective theory which gives us the exact exponents of the Heisenberg model. To this end the following Hubbard–Stratonovich transformation is convenient

$$H = \frac{1}{2} \sum_{\langle xy \rangle} (B^{ij}_{xy} C^{\dagger}_{\alpha i x} C_{\alpha j y} + \text{HC} + \bar{B}^{ji}_{yx} B^{ij}_{xy}) \tag{2.5}$$

where B_{xy} is a $2S \times 2S$ auxiliary matrix. Applying the equation of motion for B_{xy} to the Hamiltonian (2.5), we can go back to equation (2.4)†. In the mean-field approximation, B_{xy} is a constant matrix and we require only a constant $SU(2S) \times U(1)$ gauge invariance. The mean-field Hamiltonian H_0 can be diagonalized in the Fourier transformation:

$$H_0 = \frac{1}{2} \sum_{-(\pi/a) \leqslant k \leqslant (\pi/a)} [(B^{ij} + \bar{B}^{ji}) \cos ka + \mathrm{i}(B^{ij} - \bar{B}^{ji}) \sin ka] C^{\dagger}_{\alpha i}(k) C_{\alpha j}(k) \tag{2.6}$$

where a is a lattice-spacing parameter. The second term in equation (2.6) must vanish by parity symmetry $C(k) \to C(-k)$ and thus B is a Hermitian matrix. Furthermore B^{ij} can be written as $B_0 \delta^{ij}$ by the $SU(2S)$ invariance, then the diagonalized Hamiltonian H_0 is

$$H_0 = B_0 \sum_k \cos ka C^{\dagger}_{\alpha i}(k) C_{\alpha i}(k). \tag{2.7}$$

The value of B_0 is related to the Fermi velocity. The mean-field ground state has a half-filled Fermi sea which guarantees the total fermion number constrained by equation (2.2). The Fermi surface is at $\pm\pi/2a$. If one could perform the path integration over B^{ij}_{xy} and $C_{\alpha i x}$ around the solution of the mean field with the constraints equations (2.2), (2.3), all the obtained results would be exact. Since it is too demanding to perform here, we derive a suitable effective theory which can describe the low-energy phenomena of the Heisenberg model. The low-energy phenomena must be dominated by hole creation just below the Fermi surface and particle creation just above it. Thus we are only concerned with the operator $C(k)$ for k near the Fermi surface $\pm\pi/2a$. In fact we assume that the operators $C(k)$ outside the range $|k \pm \pi/2a| \leqslant \Lambda$ can be truncated from our low-energy effective theory, where Λ is an arbitrary ultraviolet cut-off parameter which is much less than the Fermi momentum. Thus we write

$$\frac{1}{\sqrt{a}} C_{\alpha i x} = \mathrm{e}^{\mathrm{i}(\pi/2a)x} \psi_{+\alpha i}(x) + \mathrm{e}^{-\mathrm{i}(\pi/2a)x} \psi_{-\alpha i}(x) \tag{2.8}$$

† In the path-integral formalism, we go back to equation (2.4) integrating over B_{xy}. Note that the Gaussian integral of B_{xy} has the 'appropriate' sign only if the original Hamiltonian is antiferromagnetic.

where $\psi_+(x)$ and $\psi_-(x)$ are slowly varying on the lattice scale. The commutation relations between ψ and $\psi^\dagger$ are $\{\psi_+(x), \psi_+^\dagger(y)\} = \delta(x-y)$, $\{\psi_-(x), \psi_-^\dagger(y)\} = \delta(x-y)$ and others vanish. We ignore the rapidly oscillating terms in any calculations. In this sense, ψ_+ and ψ_- are independent of each other at equal time. Although our derivation is not rigorous, we can check the derived effective theory by the Bethe ansatz solution for the spin-$\frac{1}{2}$ case. After checking the results, we can calculate various quantities in our effective theory. Furthermore, we may apply our consideration to the higher spin case. It is not rigorous but plausible and useful. Note the chiral Z_2 symmetry $\psi_+ \to -\psi_+$ originates from the translational symmetry $C_x \to iC_{x+a}$ in the Heisenberg model. Constraints (2.2) and (2.3) can be represented in terms of ψ_+ and ψ_- as follows.

$$\psi^\dagger_{+\alpha i}\psi_{+\alpha i} + \psi^\dagger_{-\alpha i}\psi_{-\alpha i} = \text{constant} \tag{2.9}$$

$$\psi^\dagger_{+\alpha i}\tau^a_{ij}\psi_{+\alpha j} + \psi^\dagger_{-\alpha i}\tau^a_{ij}\psi_{-\alpha j} = 0 \tag{2.10}$$

$$\psi^\dagger_{+\alpha i}\psi_{-\alpha j} + \psi^\dagger_{-\alpha i}\psi_{+\alpha j} = 0. \tag{2.11}$$

Equations (2.9) and (2.10) express the constraint for vector $U(1)$ and $SU(2S)$ currents, respectively.

Next, we consider the integration of the the B_{xy} field in the long-wave approximation. In order to take a naive continuum limit, we parametrize B_{xy} as

$$\begin{aligned} B_{xy} &= B_0 e^{aV_{xy}} \simeq B_0(1 + aV_{xy}) \\ B^\dagger_{yx} &= B_0 e^{aV^\dagger_{yx}} \simeq B_0(1 + aV^\dagger_{yx}) \end{aligned} \tag{2.12}$$

and define

$$A_1 \equiv \tfrac{1}{2}(V_{xy} - V^\dagger_{yx}) \qquad R_{xy} \equiv \tfrac{1}{2}(V_{xy} + V^\dagger_{yx}). \tag{2.13}$$

Note that in the long-wave approximation A_{xy} transforms as

$$\delta A_1 = -\partial_1\epsilon - [A_1, \epsilon] \tag{2.14}$$

under the infinitesimal gauge transformation $\delta B_{xy} = [\epsilon, B_{xy}]$. Integration over R_{xy} in equation (2.5) can be naively performed. The resulting Hamiltonian in the long-wave approximation is

$$\begin{aligned} H = \psi_+^\dagger i\left(\frac{d}{dx} + A_1\right)\psi_+ - \psi_-^\dagger i\left(\frac{d}{dx} + A_1\right)\psi_- \\ + \lambda_1(\psi^\dagger_{+\alpha i}\psi_{-\alpha j} - \psi^\dagger_{-\alpha i}\psi_{+\alpha j})(\psi^\dagger_{-\alpha j}\psi_{+\alpha i} - \psi^\dagger_{+\alpha j}\psi_{-\alpha i}) \end{aligned} \tag{2.15}$$

where λ_1 is a certain dimensionless constant. We have ignored some irrelevant operators which do not affect the critical theory. The effective Lagrangian can be written by taking into account the constraints (2.9)–(2.11) as follows.

$$\mathcal{L} = \bar{\psi}\gamma^\mu iD_\mu\psi - \lambda_1(i\bar{\psi}_i\gamma^5\psi_j)^2 - \lambda_2(i\bar{\psi}_i\psi_j)^2 \tag{2.16}$$

where $D_\mu = \partial_\mu + A_\mu$, $(\mu = 0, 1)$. A_0 is an anti-Hermitian matrix field which is a multiplier field for the constraints (2.9) and (2.10). Constraint (2.11) can be achieved by the last term in equation (2.16) which is invariant under the chiral Z_2 transformation $\psi_+ \to -\psi_+$. Thus we can regard the coupling λ_2 to be large enough compared with the cut-off parameter Λ.

3. Strong coupling gauge theories in two dimensions

As the first step towards analysing the effective Lagrangian (2.16), we treat the four-fermion interactions as a perturbation. In this section, we consider the unperturbed Lagrangian

$$\mathcal{L}_0 = \mathrm{i}\bar{\psi}\gamma^\mu D_\mu \psi. \tag{3.1}$$

Here $U(1)$ and $SU(2S)$ gauge fields are denoted by A_μ and B_μ, respectively. So the covariant derivative is written as†

$$D_\mu \equiv \partial_\mu - \mathrm{i}A_\mu + B_\mu. \tag{3.2}$$

The Dirac field $\psi_{\alpha i}(x)$ has a colour index $i = 1, \ldots, 2S$ and a spin index $\alpha = \uparrow, \downarrow$ which is a flavour in QCD terminology. Here we show that the unperturbed model is exactly solvable and is equivalent to the level-$2S$ $SU(2)$ WZW model. In the next section, we consider our model Lagrangian (2.16) as a perturbed model of equation (3.1). The coupling constant in two-dimensional QCD has the dimension of mass, and a possible gauge theory as a critical theory only is realized in the weak or strong coupling limit. Our model can be regarded as a strong coupling limit in QCD, which is a stable fixed point against the F^2 term deformation [8]. This naive dimensional analysis is consistent with Zamolodchikov's c-theorem, which claims that a central charge of conformal field theory at an infrared fixed point is less than that at an ultraviolet fixed point. In the following, we obtain physical operators of the model (3.1) and their conformal dimensions.

3.1. Free-field representation

First we rewrite the Lagrangian (3.1) in terms of free fields. The Lagrangian (3.1) is decoupled into the left- and right-moving sectors‡:

$$\mathcal{L}_0 = \psi_+^\dagger(\mathrm{i}\partial_- + A_- + \mathrm{i}B_-)\psi_+ + \psi_-^\dagger(\mathrm{i}\partial_+ + A_+ + \mathrm{i}B_+)\psi_-. \tag{3.3}$$

The classical Lagrangian is invariant under the chiral gauge transformations which independently transform the left and right sectors.

$$\begin{aligned}
&\psi_+ \to \psi'_+ \equiv g_1 \mathrm{e}^{\mathrm{i}\Lambda_1}\psi_+ && \psi_- \to \psi'_- \equiv g_2 \mathrm{e}^{\mathrm{i}\Lambda_2}\psi_- \\
&A_+ \to A'_+ \equiv A_- + \partial_+\Lambda_2 && A_- \to A'_- \equiv A_- + \partial_-\Lambda_1 \\
&B_+ \to B'_+ \equiv g_2 B_+ g_2^{-1} + g_2\partial_+ g_2^{-1} && B_- \to B'_- \equiv g_1 B_- g_1^{-1} + g_1\partial_- g_1^{-1}.
\end{aligned} \tag{3.4}$$

However, no regularization can preserve all the local symmetries due to the chiral anomaly. Here we respect vector local transformations defined by setting $\Lambda_1 = \Lambda_2$, $g_1 = g_2$ in the above chiral transformations (3.4).

The partition function Z of the model (3.1) is

$$Z = \int \frac{\mathcal{D}(\text{Fields})}{V_{\text{gauge}}} \exp \mathrm{i} \int \mathrm{d}^2x\, \mathcal{L}_0 \tag{3.5}$$

† The non-Abelian gauge fields B_μ are expanded as $B_\mu^a \tau^a$ where τ^a is an anti-Hermitian matrix in the fundamental representation of the $su(2S)$ algebra satisfying $[\tau^a, \tau^b] = f^{abc}\tau^c$, $\mathrm{Tr}\tau^a\tau^b = -1/2\delta^{ab}$.

‡ We adopt the following notation: $\gamma^0 = \sigma^1$, $\gamma^1 = i\sigma^2$, $\psi^\dagger = (\psi_+^\dagger, \psi_-^\dagger)$, $\partial_\pm = \partial_0 \pm \partial_1$.

where $V_{\rm gauge}$ eliminates redundant integrals originating from gauge symmetry. It is convenient to parametrize the gauge fields in the following way

$$A_+ = \partial_+\phi \qquad B_+ = -\partial_+ g g^{-1}$$
$$A_- = \partial_-\varphi \qquad B_- = -\partial_- h h^{-1}. \tag{3.6}$$

The change of variables induces a Jacobian in the partition function. We can exponentiate it by employing the Grassmannian fields (i.e. Faddeev–Popov ghosts).

$$\mathcal{D}A_+\mathcal{D}A_-\mathcal{D}B_+\mathcal{D}B_- = \mathcal{D}\phi\mathcal{D}\varphi\mathcal{D}g\mathcal{D}h \det\partial_+ \det\partial_- \det\nabla_+ \det\nabla_-$$
$$= \mathcal{D}\phi\mathcal{D}\varphi\mathcal{D}g\mathcal{D}h \exp\left(\mathrm{i}\int \mathrm{d}^2x\, \bar{\gamma}\mathrm{i}\partial_+\bar{\beta} + \gamma\mathrm{i}\partial_-\beta - 2\,\mathrm{Tr}\,\bar{b}\mathrm{i}\nabla_+\bar{c} - 2\,\mathrm{Tr}\,b\mathrm{i}\nabla_- c\right) \tag{3.7}$$

where $\nabla_\mu c \equiv \partial_\mu + [B_\mu, c]$. The measure in the space of SU(2S) group-valued functions is formally defined by

$$\mathcal{D}g \equiv \prod_x \mathrm{d}g(x)g^{-1}(x) \qquad g \in SU(2S). \tag{3.8}$$

Then we carry out the vector gauge transformation generating the following change

$$A'_\mu = A_\mu - \partial_\mu\varphi \qquad B'_\mu = h^{-1}B_\mu h + h^{-1}\partial_\mu h. \tag{3.9}$$

After the gauge transformation, g changes to $g' = h^{-1}g$. The measure of the partition function is invariant under the vector gauge transformation in our regularization. The resulting Lagrangian does not contain φ and h, and $\mathcal{D}\varphi\mathcal{D}h$ completely cancel out with the gauge volume $V_{\rm gauge}$.

The next step is to remove A_+ and B_+ by the chiral rotation

$$\tilde{\psi}_- = g'^{-1}\mathrm{e}^{-\mathrm{i}\phi'}\psi'_- \tag{3.10}$$

$$\tilde{\bar{b}} = g'^{-1}\bar{b}'g' \qquad \tilde{\bar{c}} = g'^{-1}\bar{c}'g'. \tag{3.11}$$

The measures of the Grassmannian fields are not invariant under a chiral rotation.

The response W from the transformation (3.10) is calculated from the following relation.

$$\int \mathcal{D}\psi'^{\dagger}_-\mathcal{D}\psi'_- \exp\left[\mathrm{i}\int \mathrm{d}^2x\, \psi'^{\dagger}_-\mathrm{i}D_+\psi'_-\right] = \int \mathcal{D}\tilde{\psi}^{\dagger}_-\mathcal{D}\tilde{\psi}_- \exp\left[\mathrm{i}\int \mathrm{d}^2x\, \tilde{\psi}^{\dagger}_-\mathrm{i}\partial_+\tilde{\psi}_- + \mathrm{i}W\right]. \tag{3.12}$$

Namely

$$\mathrm{i}W = \ln\det D_+. \tag{3.13}$$

The Dirac determinant on the right-hand side is calculated in [9]. One finds

$$W = -2W_{\rm WZW}[g'] - \frac{S}{2\pi}\int \mathrm{d}^2x\, \partial_+\phi'\partial_-\phi' \tag{3.14}$$

where $W_{\rm WZW}[g']$ is the WZW functional defined by

$$W_{\rm WZW}[g] \equiv -\frac{1}{8\pi}\left(\int \mathrm{d}^2x\, \partial_+ gg^{-1}\partial_- gg^{-1} + \frac{2}{3}\int \mathrm{d}^2x\, \mathrm{d}s\, \epsilon^{abc}\partial_a gg^{-1}\partial_b gg^{-1}\partial_c gg^{-1}\right). \tag{3.15}$$

Similarly we can calculate the response to the change (3.11) regarding ∇_+ as the Dirac operator to the adjoint representation:

$$\mathcal{D}\bar{b}'\mathcal{D}\bar{c}' = \mathcal{D}\bar{\tilde{b}}\mathcal{D}\bar{\tilde{c}}\exp(-\mathrm{i}2C_v W_{\rm WZW}[g']) \tag{3.16}$$

where $f^{abc}f^{dbc} \equiv C_v\delta^{ad}$. Thus we obtain the free-field realization of the partition function (3.5) (hereafter we omit primes)

$$Z = \int \mathcal{D}(\text{Fields})\exp[\mathrm{i}(S_{\rm matter} + S_{\rm gauge} + S_{\rm ghost})] \tag{3.17}$$

where

$$\begin{aligned} S_{\rm matter} &= \int \mathrm{d}^2x(\psi_+^\dagger \mathrm{i}\partial_-\psi_+ + \tilde{\psi}_-^\dagger \mathrm{i}\partial_+\tilde{\psi}_-) \\ S_{\rm gauge} &= -\frac{S}{2\pi}\int \mathrm{d}^2x\, \partial_+\phi\partial_-\phi - (2+2C_v)W_{\rm WZW}[g] \\ S_{\rm ghost} &= -\int \mathrm{d}^2x(\bar{\gamma}\mathrm{i}\partial_+\bar{\beta} + \gamma\mathrm{i}\partial_-\beta - 2\,\mathrm{Tr}\,\bar{\tilde{b}}\mathrm{i}\partial_+\bar{\tilde{c}} - 2\,\mathrm{Tr}\,b\mathrm{i}\partial_- c). \end{aligned} \tag{3.18}$$

Although the kinetic terms of the bosonic fields in the gauge sector have a minus sign, the physical space is positive definite [10]: the bosonic fields in the gauge sector cannot appear in the physical space by themselves.

The central charge of the model (3.1) comes from three sectors appearing in the action (3.18). The matter sector consists of $4S$ Dirac fermions, so that the central charge of this sector $c_{\rm matter}$ is

$$c_{\rm matter} = 4S. \tag{3.19}$$

The gauge sector has a single boson and a level $-(2+2C_v)$ $SU(2S)$ WZW field. Therefore the central charge of this sector is [11]

$$c_{\rm gauge} = 1 + \frac{(-2-2C_v)\dim SU(2)}{(-2-2C_v)+C_v} = \frac{4S^2(2S+1)-S}{S+1}. \tag{3.20}$$

The central charge of the ghost sector $c_{\rm ghost}$ can be obtained from the two-point functions of the energy–momentum tensor (EM tensor) of the ghosts. The result is

$$c_{\rm ghost} = -2 - 2C_2 = -8S^2. \tag{3.21}$$

In this way, one finds that the total central charge of the model is

$$c_{\rm total} = c_{\rm matter} + c_{\rm gauge} + c_{\rm ghost} = 3S/(S+1). \tag{3.22}$$

This is identical to that of the level-$2S$ $SU(2)$ WZW model.

We can show that the EM tensor of the action (3.18) is identical to that of the level-$2S$ $SU(2)$ WZW model [12]. The EM tensor of the matter sector is written in a summation of quadratic forms of a colour current, a flavour current and a $U(1)$ current [13]. However, the EM tensors of the colour part and of the $U(1)$ part do not contribute to the physics because they are, together with the EM tensors of the gauge sector and the ghost sector, written in BRST exact form. The remaining EM tensor, which is the quadratic form of the flavour current, is exactly same as that of the level-$2S$ $SU(2)$ WZW model.

Next we shall clarify the physical operators of the model (3.1) in order to see the relationship with the WZW model in detail.

3.2. Physical observables

We consider the meson-like operator $S_{\alpha,\beta}(x)$ defined as

$$S_{\alpha,\beta}(x) \equiv \psi^{\dagger}_{i\alpha+}(x)\psi_{i\beta-}(x). \tag{3.23}$$

It is represented by the free field using equations (3.10) and (3.11)

$$S_{\alpha,\beta}(x) = \psi^{\dagger}_{\alpha i+}(x)g_{ik}(x)\mathrm{e}^{\mathrm{i}\phi(x)}\tilde{\psi}_{\beta k-}(x). \tag{3.24}$$

This belongs to the fundamental representation of the chiral $su(2)\times su(2)$ algebra. We can compute the conformal dimension of $S_{\alpha,\beta}(x)$, $(\Delta_{1/2}, \bar{\Delta}_{1/2})$, collecting the dimensions of the free fields in the right-hand side of equation (3.24). The free Dirac field ψ_+ ($\tilde{\psi}_-$) has the conformal dimension $(\frac{1}{2}, 0)$ ($(0, \frac{1}{2})$). According to Knizhnik and Zamolodchikov [11], a level-k $SU(N)$ WZW field $h_{\alpha\beta}$ has the conformal dimension

$$\left(\frac{C_2}{k+C_v}, \frac{C_2}{k+C_v}\right) \tag{3.25}$$

where $C_2 = (N^2-1)/2N$. Therefore level-$(-2-2S)$ $SU(2S)$ WZW field g_{ik} in the right-hand side of equation (3.24) has the dimension

$$\left(-\frac{4S^2-1}{8S(S+1)}, -\frac{4S^2-1}{8S(S+1)}\right). \tag{3.26}$$

The dimension of $\mathrm{e}^{\mathrm{i}\phi(x)}$ is directly calculated from the two-point function

$$\begin{aligned}\langle \mathrm{e}^{\mathrm{i}\phi(x)}\mathrm{e}^{-\mathrm{i}\phi(y)}\rangle &= \text{constant}\exp\left[\frac{1}{4S}\ln(x^+-y^+)(x^--y^-)\right]\\ &= \text{constant}(x^+-y^+)^{1/4S}(x^--y^-)^{1/4S}.\end{aligned} \tag{3.27}$$

This indicates $\mathrm{e}^{\mathrm{i}\phi}$ has the dimension

$$\left(-\frac{1}{8S}, -\frac{1}{8S}\right). \tag{3.28}$$

Hence, the conformal dimension of $S_{\alpha,\beta}$ is

$$\Delta_{1/2} = \bar{\Delta}_{1/2} = \frac{1}{2} - \frac{4S^2-1}{8S(S+1)} - \frac{1}{8S} = \frac{3/4}{2S+2}. \tag{3.29}$$

This result implies that $S_{\alpha,\beta}$ is identical to the fundamental field of the level-$2S$ $SU(2)$ WZW field $h_{\alpha\beta}$.

In order to check this identification, let us construct a primary field with higher chiral $su(2)$ spin. Define

$$S_{\alpha_1\cdots\alpha_{2j},\beta_1\cdots\beta_{2j}} \equiv \mathrm{Sym}(S_{\alpha_1,\beta_1}\cdots S_{\alpha_{2j},\beta_{2j}}) \tag{3.30}$$

where $\mathrm{Sym}(\cdots)$ means to symmetrize the indices $(\alpha_1\cdots\alpha_{2j})$ and $(\beta_1\cdots\beta_{2j})$. So $S_{\alpha_1\cdots\alpha_{2j},\beta_1\cdots\beta_{2j}}$ has the chiral $su(2)$ spin (j, j), where $j \in Z/2$. From equation (3.24), we see that the indices $(i_1, \ldots, i_{2j})$ and $(k_1, \ldots, k_{2j})$ of the product $g_{i_1k_1} \cdots g_{i_{2j}k_{2j}}$ are antisymmetrized in $S_{\alpha_1,\ldots,\alpha_{2j},\beta_1,\ldots,\beta_{2j}}$. Therefore $g_{i_1k_1} \cdots g_{i_{2j}k_{2j}}$ belongs to the $2j$-antisymmetric representation of chiral $su(2S)$ algebra and has the conformal dimension

$$\left(-\frac{(2S+1)j(S-j)/S}{2+2S}, -\frac{(2S+1)j(S-j)/S}{2+2S}\right). \tag{3.31}$$

The dimension of $S_{\alpha_1\cdots\alpha_{2j},\beta_1\cdots\beta_{2j}}$, $(\Delta_j, \bar{\Delta}_j)$, is calculated using the result (3.31) and the dimensions of $(\psi_+\bar{\psi}_-)^{2j}$ and $\mathrm{e}^{\mathrm{i}2j\phi}$:

$$\Delta_j = \bar{\Delta}_j = \frac{2j}{2} - \frac{\frac{(2S+1)j(S-j)}{S}}{2+2S} - \frac{(2j)^2}{8S} = \frac{j(j+1)}{2S+2}. \tag{3.32}$$

This is precisely equal to the dimension of $\mathrm{Sym}(h_{\alpha_1\beta_1} \cdots h_{\alpha_{2j}\beta_{2j}})$. We notice the fact that the Pauli principle restricts the range of j to $0 \leqslant j \leqslant S$, which is also an expected result [11].

Another kind of a gauge-invariant operator is a flavour current

$$\boldsymbol{J}_\pm(x) \equiv \psi^\dagger_{\pm i\alpha}(x)(\boldsymbol{\sigma}_{\alpha\beta}/2)\psi_{\pm i\beta}(x) \tag{3.33}$$

where the σ^a are Pauli matrices. They satisfy the level-$2S$ Kac–Moody algebra. The whole physical space of the model is spanned by acting Fourier modes of $\boldsymbol{J}_\pm(x)$ to some states generated by a product of the operator $S_{\alpha,\beta}$. Therefore we can conclude that the physical Hilbert space is equivalent to the Hilbert space of the level-$2S$ $SU(2)$ WZW model.

4. Perturbation theory in a strong coupling QED for the $S = \frac{1}{2}$ Heisenberg model

Our goal is to solve the model given by equation (2.16). As explained in the previous section, this model at $\lambda_1 = \lambda_2 = 0$ is described in the level-$2S$ $SU(2)$ WZW model, and thus we consider perturbed theories of the WZW model with the couplings λ_1 and λ_2. In the Lagrangian (2.16), the chiral symmetry is violated apparently by the two independent coupling constants λ_1 and λ_2. Since the chiral symmetry is generally believed to protect massless excitations in field theory, we have to answer why the massless excitations survive in the Heisenberg model. First, we discuss the $S = 1/2$ case and check the results by the finite-size scaling in the Bethe ansatz solution. After that we discuss other spin cases in the next section.

In the $S = 1/2$ case, the model (2.16) is a two-flavour strong coupling $U(1)$ gauge theory with four-fermion interactions. For the description in terms of the level-1 $SU(2)$ WZW model, the following expression for the four-fermion interaction is more manifestly written

$$\begin{aligned} -\lambda_1(\bar{\psi}_\alpha \mathrm{i}\gamma^5\psi_\alpha)^2 - \lambda_2(\bar{\psi}_\alpha\psi_\alpha)^2 &= 4(\lambda_1+\lambda_2)\boldsymbol{J}_+\cdot\boldsymbol{J}_- + (\lambda_1+\lambda_2)j_+j_- \\ &\quad + \tfrac{1}{2}(\lambda_1-\lambda_2)(\epsilon^{\alpha\beta}\psi^\dagger_{\alpha+}\psi^\dagger_{\beta+}\epsilon^{\gamma\delta}\psi_{\gamma-}\psi_{\delta-} + \mathrm{HC}). \end{aligned} \tag{4.1}$$

Note that the first and last terms violate chiral $SU(2)$ and $U(1)$ symmetry, respectively. The last term does not contain any $SU(2)$ degrees of freedom but a $U(1)$ degree of freedom, which is killed by the $U(1)$ gauge field. In fact, the last term is a dimension zero operator which is just constant and affects nothing in the physics. The second term on the right-hand side of equation (4.1) vanishes because of the constraints (2.9). We have to take into account only the first term in the right-hand side of equation (4.1) which has the conformal dimension $(1, 1)$. The $SU(2)$ excitations are independent of the $U(1)$ degree of freedom. Thus the system is described in the level-1 $SU(2)$ WZW model with a perturbation $J_+ \cdot J_-$. Although this term violates the chiral $SU(2)$ symmetry, it is marginally irrelevant for a positive coupling constant $\lambda_1 + \lambda_2 > 0$. Since λ_2 should be large enough to realize the constraint in our case, the perturbation $J_+ \cdot J_-$ must vanish in the low-energy limit. This is the reason why the spin-$\frac{1}{2}$ Heisenberg model has massless excitations despite the existence of the chiral symmetry violating interactions. This obtained effective theory for the Heisenberg model is identical to the continuum field theory model derived from the half-filled Hubbard model by Affleck and Haldane [3].

We can check whether this continuum field theory describes the low-energy physics of the spin-$\frac{1}{2}$ Heisenberg model by the finite-size scaling of the Bethe ansatz solution. This work has already been done by Affleck *et al* [4, 5]. The finite-size effect in the energy of the ground and excited states is calculated in conformal field theory by a conformal mapping from a complex plane onto a cylinder with a finite circumference L [14]. The leading term of the ground-state energy is proportional to L. The L^{-1} correction to the ground-state energy gives us the central charge of the system. An excitation energy proportional to L^{-1} gives us the conformal dimension of the corresponding primary field. If we study these finite-size effects in the solution of a lattice model (for example the Bethe ansatz solution of the Heisenberg model), we find the corresponding conformal field theory with suitable central charge and a whole set of primary fields which describe the critical phenomena in the model. Generally, if the conformal field theory has marginally irrelevant operators there are further logarithmic corrections to the finite-size effects on the energy levels of the model [15]. The ground-state energy has a $L^{-1}(\log L)^{-3}$ correction and the excited energy has an $L^{-1}(\log L)^{-1}$ correction which can be calculated within the framework of a perturbation of the conformal field theory.

The coefficients of these corrections are determined by the three-point functions of the operators. If there are logarithmic corrections to the lattice model, the low-energy phenomena of the system can be described in the deformed conformal field theory with certain marginally irrelevant perturbations.

In the spin-$\frac{1}{2}$ Heisenberg model, we can compare the results of the Bethe ansatz solution to the level-1 $SU(2)$ WZW model with the $J_+ \cdot J_-$ perturbation. The result is good as we will show below.

The single-loop calculation in the WZW model shows that the sign of the bare coupling constant determines whether the $J_+ \cdot J_-$ perturbation is marginally relevant or irrelevant. In our case $|\lambda_1| < \lambda_2$, $J_+ \cdot J_-$ is marginally irrelevant, and thus we can calculate the finite-size effects in the perturbation theory. The finite-size correction of the ground-state energy E_0 to the bulk part $\epsilon_0 L$ is

$$E_0 - \epsilon_0 L = -\frac{\pi}{6L}\left(c + \frac{d_0}{(\log L)^3}\right). \tag{4.2}$$

The energy gap of the first excited states are given by

$$E_t - E_0 = \frac{\pi^2}{L}\left(x_t + \frac{d_t}{\log L}\right) \tag{4.3}$$

$$E_s - E_0 = \frac{\pi^2}{L}\left(x_s + \frac{d_s}{\log L}\right) \tag{4.4}$$

where E_t and E_s are the energy of triplet and singlet excitations, respectively. We can calculate c, x_t, x_s, d_0, d_t, and d_s both in the WZW model and the Bethe ansatz solution as follows.

	c	x_t	x_s	d_0	d_t	d_s
WZW	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$-\frac{1}{4}$	$\frac{3}{4}$
Bethe ansatz	1	$\frac{1}{2}$	$\frac{1}{2}$	0.3433	$-\frac{1}{4}$	$\frac{3}{4}$.

(4.5)

The results of the Bethe ansatz solution were obtained by Woynarovich and Eckle [16]. They agree with each other except for d_0. This reason for the disagreement in d_0 has not yet been understood.

On the basis of the finite-size scaling analysis of the Bethe ansatz solution, we conclude that low-energy phenomena in the spin-$\frac{1}{2}$ Heisenberg model can be described in the level-1 $SU(2)$ WZW model with the marginally irrelevant perturbation $J_+ \cdot J_-$. Various quantities can be calculated within this framework from now.

5. A relevant perturbation in the WZW model for the $S > \frac{1}{2}$ Heisenberg model

In the $S > 1/2$ case, we have to reconsider the four-fermion interaction within the framework of the level-$2S$ $SU(2)$ WZW model. The meson-like operator $\psi^\dagger_{\alpha i+}\psi_{\beta i-}$ can be identified with the $SU(2)$ valued field $h_{\alpha\beta}(x^+, x^-)$ in the WZW model, which has conformal dimensions

$$\left(\frac{3}{8S+8}, \frac{3}{8S+8}\right). \tag{5.1}$$

The four-fermion interaction $(\bar\psi\psi)^2$ is a relevant operator $(\mathrm{Tr}\,h)^2$ which has conformal dimensions

$$\left(\frac{1}{S+1}, \frac{1}{S+1}\right) \tag{5.2}$$

in the WZW model. The essential difference between the $S > 1/2$ case and the $S = 1/2$ case is the existence of the relevant operator with chiral Z_2 symmetry $\psi_+ \to -\psi_+$ (or $h_{\alpha\beta} \to -h_{\alpha\beta}$ in terms of the WZW model). There is no chiral Z_2 symmetric relevant operator in the unitary representation of the $SU(2)_1$ WZW model for the $S = 1/2$ case. There are primaries with conformal dimensions $(j(j+1)/2S+2, j(j+1)/2S+2), 0 \leqslant j \leqslant S$ in the level-$2S$ $SU(2)$ WZW model. Since the primary with a half-odd-integer j has no chiral Z_2 symmetry, the only marginal perturbation $J_+ \cdot J_-$ is possible in the $S = 1/2$ case.

In the $S > 1/2$ case, the most relevant perturbation $(\mathrm{Tr}\,h)^2$ should be added to the Lagrangian of the WZW model in order to achieve the constraint $\bar\psi_{\alpha i+}\psi_{\alpha j-} = 0$. Conformal field theory with the relevant perturbation must be deformed to another fixed point or a massive theory. The $SU(2)$ WZW model with the constraint $\mathrm{Tr}\,h = 0$ is classified in two universality classes of half-odd-integer S and integer S. This can be explained as follows. At large S, we can treat the WZW model semiclassically. The $SU(2)$ group valued field $h_{\alpha\beta}$ in the WZW model can be parametrized by the two complex variables z_1, z_2 with a constraint $|z_1|^2 + |z_2|^2 = 1$ as follows

$$h = \begin{pmatrix} z_1 & -\bar z_2 \\ z_2 & \bar z_1 \end{pmatrix}. \tag{5.3}$$

If we require the constraint $\mathrm{Tr}\, h = 0$, the real part of z_1 vanishes and the field h is in a two-dimensional sphere with a radius 1. This implies that the target space of the WZW nonlinear sigma model with the constraint $\mathrm{Tr}\, h = 0$ becomes a two-dimensional sphere which is the same as the target space of the $O(3)$ symmetric nonlinear sigma model. In a practical calculation, it was confirmed that the kinetic term and the WZ term of the constrained WZW model are identical to the kinetic term and θ term at $\theta = 2S\pi$ of the $O(3)$ nonlinear sigma model, respectively. We obtained this model for the Heisenberg model in a different way. The θ term discriminates between the models half-odd-integers and integers. Since the $O(3)$ nonlinear sigma model is asymptotically free and the coupling constant is proportional to S^{-2} in our situation, the renormalization-group flow with increasing length is running toward smaller S within one universality class. This is consistent with Zamolodchikov's c-theorem in the deformation of the level-$2S$ $SU(2)$ WZW model. The c-function takes the value $6S/(2S+2)$ at a fixed point which is described by the level-$2S$ $SU(2)$ WZW model. The renormalization flow is running toward a model with a smaller value of c-function within one universality class. In the half-odd integer spin universality class, only the $S = 1/2$ model has a stable fixed point which is described in the $SU(2)_1$ WZW model. Other fixed points in the half-odd-integer spin case are unstable against a $(\mathrm{Tr}\, h)^2$ perturbation and might be deformed into the $SU(2)_1$ WZW model. Thus, the low-energy phenomena of the half-odd-integer spin Heisenberg model can be described by the $SU(2)_1$ WZW model with the marginally irrelevant perturbation $J_+ \cdot J_-$. In the integer spin universality class, there is no stable fixed point. Thus the field theory becomes massive. Our consideration is plausible despite the semiclassical view point of the universality classification. The Bethe ansatz and the Affleck–Kennedy–Leib–Tasaki (AKLT) solutions [2] agree with the continuum field theory at $S = 1/2$ and $S = 1$ where the quantum effects are very strong.

6. Summary

We have derived 2D continuum gauge theories as effective theories of the 1D quantum Heisenberg antiferromagnets with arbitrary spin S. In order to represent the spin matrix in terms of fermion operators, local constraints are necessary as in continuum field theories. The $SU(2S) \times U(1)$ current constraints can be realized in terms of strong coupling gauge theories and other constraints can be achieved by adding a certain perturbation term. $SU(2S) \times U(1)$ degrees of freedom are confined by the gauge field and it was shown that unperturbed gauge theory is equivalent to the level-$2S$ $SU(2)$ WZW model.

We show that the low-energy phenomena in the spin-$\frac{1}{2}$ Heisenberg model can be described in the level-1 $SU(2)$ WZW model with a marginally irrelevant perturbation $J_+ \cdot J_-$ which violates the chiral $SU(2)$ symmetry. Since this perturbation vanishes at the critical point, the massless excitations can survive despite the violation of the chiral $SU(2)$ symmetry. Several quantities of this continuum field theory were checked by the Bethe ansatz solution in the finite-size scaling method. The results are good.

We can apply our derivation of an effective Lagrangian to some extended antiferromagnetic chain models with higher spin [6, 7]. They are solved using the Bethe ansatz and it is believed that these solvable models have gapless excitations. This has been confirmed in several practical models using analytic and numerical methods [4, 17]. Here, we consider the $S = 1$ case in terms of continuum field theory. The model described by the following bilinear–biquadratic Hamiltonian

$$H = \sum_{\langle x,y\rangle} [S_x \cdot S_y - \beta (S_x \cdot S_y)^2] \tag{6.1}$$

is solvable at $\beta = 1$ and -1 using the Bethe ansatz [6,7]. At $\beta = -1$, the model possesses an $SU(3)$ symmetry and a gapless excitation. To see this, we introduce three fermion operators $C_{\alpha x}$ [13]. The spins can be written:

$$S^a{}_x = C^\dagger{}_{\alpha x} L^a{}_{\alpha\beta} C_{\beta x} \tag{6.2}$$

where the L^as are the $S = 1$ generators. The states for the spin system satisfy the local constraint at each lattice site as in the $S = 1/2$ case.

$$C^\dagger{}_{\alpha x} C_{\alpha x} |\text{phys}\rangle = 2|\text{phys}\rangle. \tag{6.3}$$

The relations

$$L^a{}_{\alpha\beta} L^a{}_{\gamma\epsilon} = \delta_{\beta\gamma}\delta_{\alpha\epsilon} - \delta_{\alpha\gamma}\delta_{\beta\epsilon} \qquad (L^a L^b)_{\alpha\beta}(L^a L^b)_{\gamma\epsilon} = \delta_{\alpha\beta}\delta_{\gamma\epsilon} + \delta_{\alpha\gamma}\delta_{\beta\epsilon} \tag{6.4}$$

enable us to rewrite the hamiltonian (6.1) in terms of the fermion operators.

$$H = \sum_{\langle x,y\rangle} [-C^\dagger{}_{\alpha x} C_{\alpha y} C^\dagger{}_{\beta y} C_{\beta x} - (\beta + 1) C^\dagger{}_{\alpha x} C_{\beta x} C^\dagger{}_{\alpha y} C_{\beta y}]. \tag{6.5}$$

At $\beta = -1$, the hamiltonian has a global $SU(3)$ symmetry in addition to the local $U(1)$ gauge invariance. The low-energy phenomena in this system can be described in a strong coupling $U(1)$ gauge theory with $SU(3)$ flavour as in the case of the $S = 1/2$ Heisenberg model. In mean-field theory, the fermion momentum becomes $\pi/(3a)$, which is determined by the constraint (6.3) on the number of fermions at each lattice site. The low-energy phenomena are dominated only by those fermion operators ψ_+ and ψ_- near the Fermi surface:

$$C_{\alpha x} = \sqrt{a}(\psi_{\alpha +}(x) \mathrm{e}^{\mathrm{i}(\pi/3a)x} + \psi_{\alpha -}(x) \mathrm{e}^{-\mathrm{i}(\pi/3a)x}). \tag{6.6}$$

The effective Lagrangian of the $U(1)$ gauge theory with $SU(3)$ flavour becomes

$$\mathcal{L} = \mathrm{i}\bar{\psi}_\alpha \gamma^\mu D_\mu \psi_\alpha + (\text{marginal operators}). \tag{6.7}$$

This effective theory is equivalent to the level-1 $SU(3)$ WZW model with a certain marginal perturbation. Although this model has one relevant operator $\bar{\psi}_\alpha \psi_\alpha$ with gauge invariance, the chiral Z_3 symmetry forbids it to appear in the Lagrangian (6.7) and protects the gapless excitation as in the $S = 1/2$ Heisenberg model. The translational symmetry of one lattice bond in the original lattice model (6.1) is represented as a chiral Z_3 symmetry $\psi_+ \to \mathrm{e}^{\mathrm{i}(2\pi/3)}\psi_+$, $\psi_- \to \psi_-$ in the effective gauge theory. This chiral Z_3 symmetry forbids the relevant operator $\bar{\psi}_\alpha \psi_\alpha$, and then the gapless excitation can survive at $\beta = -1$. On the other hand, around $\beta = -1$ there is no symmetry to forbid the relevant operator, and therefore there is no reason for the model to have gapless excitations.

At the other solvable point $\beta = 1$, the existence of a gapless excitation has been verified [4,7,17]. The finite-size scaling analysis tells us that the low-energy phenomena can be described in terms of the level-2 WZW model [4,17]. In our gauge theory description, this model corresponds to an $SU(2) \times SU(2)$ Dirac fermion coupled to an $SU(2) \times U(1)$ gauge field. Since this theory has a relevant operator $(\bar{\psi}\psi)^2$ with conformal dimensions $(\frac{1}{2}, \frac{1}{2})$ which satisfies required certain symmetries, it is easily understood that a theory around $\beta = 1$ has no gapless excitation.

In other spin cases of the Heisenberg model, the system can be described by the level-$2S$ $SU(2)$ WZW model with a certain relevant perturbation, which must deform the conformal field theory to the other fixed point or massive theory. From the semiclassical view point in the level-$2S$ $SU(2)$ WZW model with the constraint $\mathrm{Tr}\, h = 0$, we see that the system becomes an $O(3)$ non-linear sigma model with the θ term at $\theta = 2\pi S$. The θ term tells us that there are two universality classes for the half-odd-integer spin and integer spin systems. The half-odd-integer spin systems can be described in the level-1 $SU(2)$ WZW model with the marginal irrelevant perturbation $J_+ \cdot J_-$ near the critical point. The integer spin systems become massive.

This is the whole story for understanding the 'Haldane conjecture' in terms of the strong-coupling gauge theory. It remains to study the occurrence of the two universality classes using suitable fermion terminology instead of bosonization.

Acknowledgments

We would like to thank I Affleck and K Hori for helpful comments. We are grateful to T Fujita and M McMullin for reading the manuscript. One of us (CI) would like to thank G Semenoff for discussions and the kind hospitality extended by the University of British Columbia.

References

[1] Haldane F D M 1983 *Phys. Rev. Lett.* **50** 1153; 1983 *Phys. Lett.* **93A** 464
[2] Affleck I, Kennedy T, Lieb E and Tasaki H 1987 *Phys. Rev. Lett.* 59 799; 1988 *Commun. Math. Phys.* **115** 477
[3] Affleck I and Haldane F D M 1987 *Phys. Rev.* B **36** 5291
[4] Affleck I, Gepner D, Shultz H J and Zimann T 1989 *J. Phys. A: Math. Gen.* **2** 511
[5] Nomura K 1992 Lagarithmic corrections of one-dimensional $S = 1/2$ Heisenberg antiferromagnet *Preprint* Tokyo Institute of Technology, TITCMT-92-1; 1993 *Phys. Rev.* B **48** 16814
[6] Sutherland B 1975 *Phys. Rev.* B **12** 3110
[7] Zamalodchikov A B and Fateev V A 1980 *Yad. Fiz.* **32** 587
Kulish P and Sklyanin E 1982 *Lecture Notes in Physics* **151** (Berlin: Springer)
Kulish P, Resfietikhin N Yu and Sklyanin E 1981 *Lett. Math. Phys.* **5** 393
Takhatajan L 1982 *Phys. Lett.* **87A** 479
Babudjian J 1982 *Phys. Lett.* **90A** 479; 1983 *Nucl. Phys.* B **215** 317
[8] Itoi C and Mukaida H 1992 *Mod. Phys. Lett.* A **7** 259
[9] Polyakov A M and Wiegmann P B 1983 *Phys. Lett.* B **131** 121
Polyakov A M 1988 *Fields, Strings and Critical Phenomena (Les Houches)* (Amsterdam: North-Holland)
[10] Gawedzki K and Kupiainen A 1990 *Nucl. Phys.* B **320** 620
[11] Knizhnik V and Zamolodchikov A B 1984 *Nucl. Phys.* B **247** 83
[12] Karabali D and Schnitzer H J 1990 *Nucl. Phys.* B **329** 649
[13] Affleck I 1986 *Nucl. Phys.* B **265** 409; 1988 *Fields, Strings and Critical Phenomena (Les Houches)* (Amsterdam: North-Holland)
[14] Blöte H W, Cardy J L and Naitingale M P 1986 *Phys. Rev. Lett.* **56** 742
[15] Cardy J L 1986 *J. Phys. A: Math. Gen.* **19** L1093
Ludwig A W W and Cardy J L 1987 *Nucl. Phys.* B **285** 687
[16] Woynarovich F and Eckle H-P 1987 *J. Phys. A: Math. Gen.* **20** L97
[17] Avdeev L V and Doerfel B-D 1987 *Teor. Mat. Fiz.* **71** 272
Avdeev L V 1990 *J. Phys. A: Math. Gen.* **23** L485
[18] Zamolodchikov A B 1986 *JETP Lett.* **43** 730
Affleck I 1986 *Phys. Rev. Lett.* **56** 746